THE QUASI-STATIC APPROXIMATION FOR CRACKED INTERFACES IN

LAYERED SYSTEMS

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## INTRODUCTION

For the last two decades the quasi-static approximation (QSA) has been the most commonly used approach for describing the interaction of ultrasonic waves with imperfect interfaces. The QSA is a low-frequency approximation and it can be used when the thickness of the interface is much smaller than the wavelength of the waves used to inspect the interface. Its most complete formulation has been presented by Baik and Thompson [1], and models the real interfacial imperfections as continuous, uniform distributions of springs and masses along the interface plane (see Fig 1).

The mathematical formulation of the QSA is provided by the modified boundary conditions enforced at the interface plane. Following Baik and Thompson [1], the QSA boundary conditions can be written as follows,

$$\frac{1}{2}\left[\sigma_{33}\left(x, z=0^{+}\right)+\sigma_{33}\left(x, z=0^{-}\right)\right]=K_{N}\left[u_{3}\left(x, z=0^{+}\right)-u_{3}\left(x, z=0^{-}\right)\right], \qquad (1.a)$$

$$\frac{1}{2}\left[\sigma_{31}\left(x, z=0^{+}\right)+\sigma_{31}\left(x, z=0^{-}\right)\right]=K_{T}\left[u_{1}\left(x, z=0^{+}\right)-u_{1}\left(x, z=0^{-}\right)\right], \qquad (1.b)$$

$$-\frac{m\omega^{2}}{2}\left[u_{3}\left(x, z=0^{+}\right)+u_{3}\left(x, z=0^{-}\right)\right]=\sigma_{33}\left(x, z=0^{+}\right)-\sigma_{33}\left(x, z=0^{-}\right), \qquad (1.c)$$

$$-\frac{m\omega^{2}}{2}\left[u_{1}\left(x, z=0^{+}\right)+u_{1}\left(x, z=0^{-}\right)\right]=\sigma_{31}\left(x, z=0^{+}\right)-\sigma_{31}\left(x, z=0^{-}\right). \qquad (1.d)$$

In these equations $K_N$ and $K_T$ are the stiffness constants of the distributed springs, and relate the discontinuity of the displacement components to the corresponding components of the stress applied to the interface. The extra-mass, $m$, models the inertial effect of the

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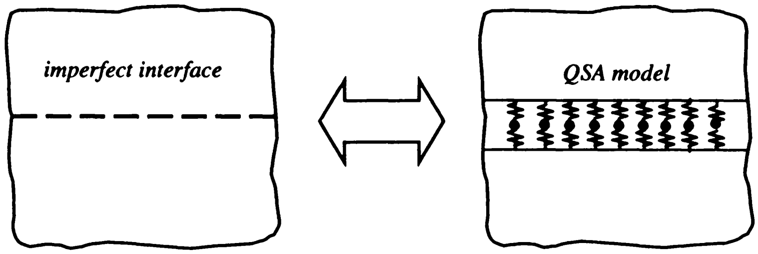


Figure 1. Imperfect interface and its model according to the QSA.

interfacial imperfections in the equilibrium condition at the interface plane. The symbol $\omega$ represents the angular frequency of the ultrasonic waves. The QSA boundary conditions are not derived from first principles. Rather they are heuristic in nature [1].

The QSA does not provide any way to correlate the values of the elastic interfacial constants $K_N$ and $K_T$ to the micro-mechanics of the defects. In reference [1] the interfacial stiffness constants are defined by the ratio between the load applied at 'infinity', $P_{N,T}$, and the extra displacement, $\Delta_{N,T}$, measured at a location far from the interface,

$$K_{N,T} = \frac{P_{N,T}}{\Delta_{N,T}}. \tag{2}$$

The extra displacement, $\Delta_{N,T}$, is not zero when the interface contains imperfections that alter its elastic properties. A few simple cases, such as that of an interphase layer embedded between two infinite half-spaces, and those of periodic distributions of one- and two-dimensional cracks, were considered in detail, and expressions for the stiffness constants were presented in terms of structural and mechanical properties of the distributed defects. Since cracks are defects with no volume, the boundary conditions at a cracked interface do not contain the inertial terms, i.e., $m = 0$.

The QSA boundary conditions (eq. (1.a-d)) have been widely applied to isolated interfaces as well as to interfaces embedded in media having a structure. Examples of such systems are layered media, fibers or whiskers embedded in a matrix, and any other system whose structure features a characteristic length. When the QSA is used to describe the elastic behavior of an imperfect interface in a system with a structure the question arises whether the system's structure affects the elastic response of the imperfect interface.

The objectives of this work are threefold. First, a derivation from first principles of the QSA boundary conditions will be presented for a randomly cracked interface. In this way, expressions for the stiffness constants that link the values of these quantities to the structural and micro-mechanical properties of the crack distribution will, thus, be obtained. Secondly, the interfacial stiffness constants of a cracked interface between a layer and its substrate will be evaluated. To this end, the crack opening displacement (COD) of an isolated interfacial crack undergoing an external uniform load will be evaluated by numerically solving a system of integral equations for the dislocation densities associated with the components of the COD. The dislocation densities are equivalent to the crack surface displacement gradients. The introduction of the crack compliance tensor connecting the COD to the applied stress will lead to the evaluation of the interfacial stiffness constants. The dependence of the latter on the layer thickness will be investigated.

Finally, examples of dispersion curves for the lowest mode supported by the layered structure will illustrate the need to account for the dependence of the stiffness constants on the structural properties of the layered system.

## QSA BOUNDARY CONDITIONS ON A CRACKED INTERFACE

In this section the cracks are assumed to be one-dimensional. However, the derivation presented here, as well as its results, can be extended to distributions of two-dimensional cracks in a straightforward manner. Consider a random distribution of cracks at the interface between two media. Let $z=0$ be the interface plane, and $u_i^+(x)$ and $u_i^-(x)$ the i-th component of the displacement just above and just below the interface, respectively. The average displacement discontinuity at the interface can be written as

$$\left\langle u_i^+(x) - u_i^-(x) \right\rangle = \left\langle \Delta u_i(x) \right\rangle = \frac{1}{L} \int_L \Delta u_i(x)\, dx, \quad i = x, z \tag{3}$$

where $L$ is the representative length of the crack distribution. The displacement discontinuity $\Delta u_i(x)$ is not zero only at the locations of the cracks, and there it is equal to the COD, $b_i(x)$. Then, equation (3) can be written as

$$\left\langle \Delta u_i \right\rangle = \frac{1}{L} \sum_{k=1}^{N} \int_{a_k} b_i(x)\, dx. \tag{4}$$

In eq. (4), $N$ is the number of cracks in $L$, and $a_k$ is the length of the k-th crack. For the sake of simplicity, let the cracks of the distribution be identical to each other. Then, equation (4) becomes,

$$\left\langle \Delta u_i \right\rangle = \nu\, a \left\langle b_i \right\rangle, \tag{5}$$

where $\nu = N/L$ is the crack density, and $\langle b_i \rangle$ is the average i-th component of the COD. The crack compliance tensor, $S_{ij}$, that relates the average displacement components to the average stress, $\langle \sigma_{ij} \rangle$, applied to the crack faces, can now be introduced,

$$\left\langle b_i \right\rangle = a\, S_{ij} \left\langle \sigma_{jz} \right\rangle n_z, \quad i, j = x, z\,. \tag{6}$$

Note that, in this equation, the normal to the interface is assumed to be parallel to the z-axis and, consequently, only the components of the stress tensor with indexes '$jz$' appear in this equation. The z-component of the unit vector $n$ is equal to 1 and, therefore, it will be omitted in the following. Introducing eq. (6) into eq. (5) the latter becomes

$$\left\langle \Delta u_i \right\rangle = \nu\, a^2\, S_{ij} \left\langle \sigma_{jz} \right\rangle. \tag{7}$$

Numerically it can be shown that the average normal (tangential) displacement component due to a uniform shear (normal) stress field is zero. Thus, only one term remains on the right hand side of eq. (7). By inverting eq. (7), the QSA boundary condition for a cracked interface can be obtained,

$$\langle \sigma_{xz} \rangle = K_{xx} \langle \Delta u_x \rangle , \tag{8.a}$$

$$\langle \sigma_{zz} \rangle = K_{zz} \langle \Delta u_z \rangle , \tag{8.b}$$

where

$$K_{xx} = K_T = \frac{1}{v\, a^2\, S_{xx}}, \quad K_{zz} = K_N = \frac{1}{v\, a^2\, S_{zz}} . \tag{9}$$

Equation (9) provides the expressions for the stiffness components of a cracked interface. It shows that $K_T$ and $K_N$ are inversely proportional to the crack density, to the square of the crack length, and to the crack compliance. Therefore, both the geometrical and the micro-mechanical properties of the interfacial defects are included in the definition of the macroscopic interfacial properties.

## CRACKED INTERFACE BETWEEN A LAYER AND A SUBSTRATE

### Isolated Interfacial Crack

In this subsection the mechanical response of an interfacial crack to an external load is examined. The system consisting of a layer of copper on a steel substrate is considered. The thickness of the layer is h.

To evaluate the COD of such a crack, a system of integral equations for the dislocation densities is solved [2]. Once the dislocation densities are known, the components of the COD can be evaluated by integrating them over the crack extension. Finally, eq. (6) is used to obtain the components of the crack compliance tensor, $S_{ij}$.

Figure 2 illustrates the behavior of the normal and tangential component of the crack compliance as a function of the ratio h/$a$. The plots show that both components tend to infinity as the layer thickness decreases, while they approach the same limit as the layer approximates a half-space. It is worth noting, also, that the normal compliance is always

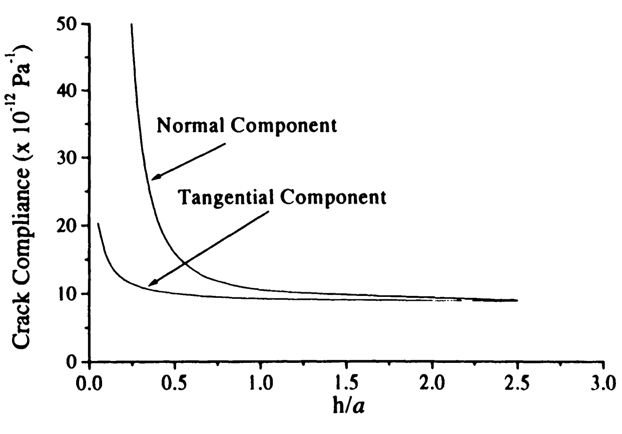


Figure 2. Compliance of crack at the interface between a copper layer and a steel substrate vs. the ratio of the layer thickness to the crack length, h/$a$.

larger than the transverse compliance. This fact can be easily understood in terms of the amount of surrounding material that opposes the crack deformation.

Interfacial Stiffness Constants

The interfacial stiffness constants can be calculated by using eq. (9). Making the assumption that the cracks do not interact with each other, and having evaluated the compliance tensor of an isolated crack as a function of the ratio h/$a$, it is possible to investigate the effect of the geometrical properties of the layered structure on the interfacial stiffness.

Figure 3 presents plots of the normal and transverse interfacial stiffness constants versus the ratio h/$a$ for three values of the crack length $a$. The normalized crack density, $a\nu$, is equal to 0.2. The plots show that the interface becomes more compliant as the layer thickness decreases, and that the larger the crack length the more compliant the interface. As for the isolated crack, both the interfacial stiffness constants approach the same limit as the thickness of the layer increases. Because of the inverse proportionality between the stiffness constants and the crack compliance, the normal stiffness is always smaller than the transverse stiffness constant. No physical interphase layer model of an imperfect interface could simulate such a property of a cracked interface.

Figure 4 shows plots of the interfacial stiffness constants versus the ratio h/$a$ for three values of the layer thickness h. The normalized crack density is again equal to 0.2. Here, the behavior of the stiffness constants is markedly different from that shown in Fig. 3. Such dependence of the stiffness constants is explained as follows. If h is constant, an increase of the ratio h/$a$ is obtained by decreasing the crack length, $a$. On the other hand, since $a\nu$ is constant, the crack density $\nu$ must be increased accordingly. Thus, as h/$a$ increases the crack distribution changes its properties becoming progressively a denser and denser distribution of very small cracks. Figure 4, therefore, shows that among different

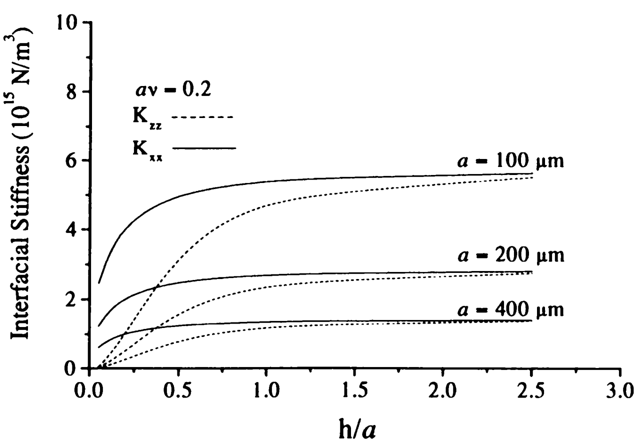


Figure 3. Interfacial stiffness constants vs. h/$a$ for three values of the crack length: $a = 100$ μm, $a = 200$ μm, $a = 400$ μm. The normalized crack density is constants, $a\nu = 0.2$.

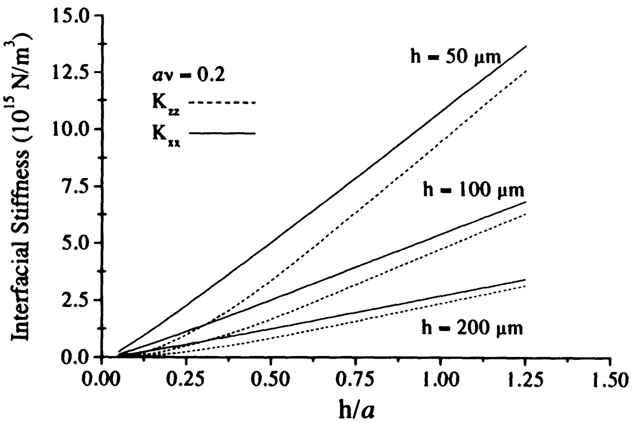


Figure 4. Interfacial stiffness constants vs. h/$a$ for three values of the layer thickness: h = 50 μm, h = 100 μm, h = 200 μm. The normalized crack density is constants, $a$v = 0.2.

interfaces with crack distributions having the same normalized crack density or, equivalently, the same cracked area, those having the smaller cracks are the stiffer. In an attempt to identify the nature of interfacial imperfections, Nagy [3] considered the ratio between the reflection coefficients of longitudinal and shear waves at normal incidence at low frequencies. He focused on interfaces between samples of the same material, and showed that the ratio $r = R_L/R_T$ can be expressed in terms of the ratio between the transverse and normal interfacial stiffness constants,

$$r = \lim_{\omega \to 0} \frac{R_L(\omega)}{R_T(\omega)} = \frac{K_T}{K_L}\frac{C_L}{C_T}. \tag{10}$$

In eq. (10), the symbols $C_L$ and $C_T$ are the phase velocity of the longitudinal and shear waves, respectively. Nagy found that $r$ is smaller than 1 for distributions of volumetric imperfections, while $r$ is larger than 1.5 for cracked interfaces. The extension of Nagy's analysis to interfaces between different materials and to structures more complex than that of an isolated interface should be straightforward. Although this is not a primary objective of this work, the behavior of the ratio $K_T/K_L$ is presented as a function of h/$a$ in Fig. 5. The plot shows that the ratio $K_T/K_N$ is always greater than one, and progressively increases as the layer thickness decreases.

## Dispersion of the lowest mode

In several applications the nondestructive assessment of the interface between a layer and its substrate is attempted by using the lowest mode supported by the system. In this section the effect of a crack distribution on the phase velocity of this mode is briefly investigated.

Figure 6 shows the relative variation of the mode's phase velocity as a function of the frequency for a system consisting of a copper layer on a steel substrate. The change of phase velocity is due to a distribution of cracks of length $a$ = 200 μm. Three values of the layer thickness are considered: $h$ = 0.15$a$, 0.4$a$, 0.65$a$. The normalized crack density is $a$v

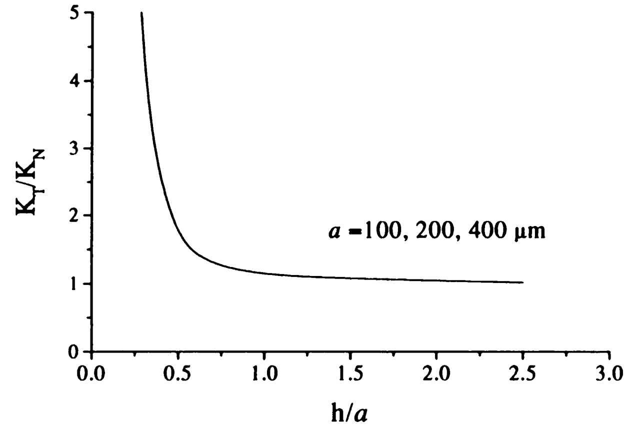


Figure 5. Stiffness ratio vs. h/$a$. The normalized crack density, $a\nu$, is equal to 0.2.

= 0.2. The reference velocity is that of the same mode propagating along the surface of a system having a perfectly bonded interface. The actual velocity is that of the mode propagating over a surface of a system having a cracked interface. The interfacial stiffness constants used to generate these plots are those presented in Fig. 3. The vertical broken line represents the upper limit for the frequency range where the QSA is expected to provide reliable results. Figure 6 shows that a frequency near 5 MHz should be used in this case in order to maximize the sensitivity of the wave to interfacial imperfections.

Figure 7 reports the percentage error in the predicted phase velocity obtained by using the values of the stiffness constants of an isolated interface between two infinite half-

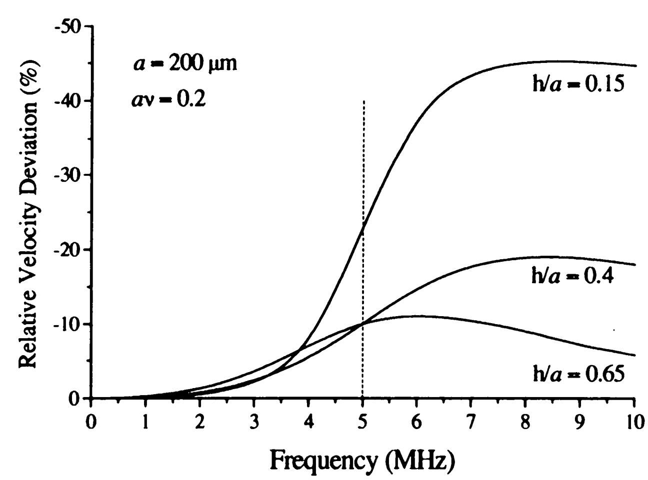


Figure 6. Relative phase velocity variation vs. angular frequency for three values of the layer thickness. The crack length is $a$ = 200 μm, and the crack density is $a\nu$ = 0.2.

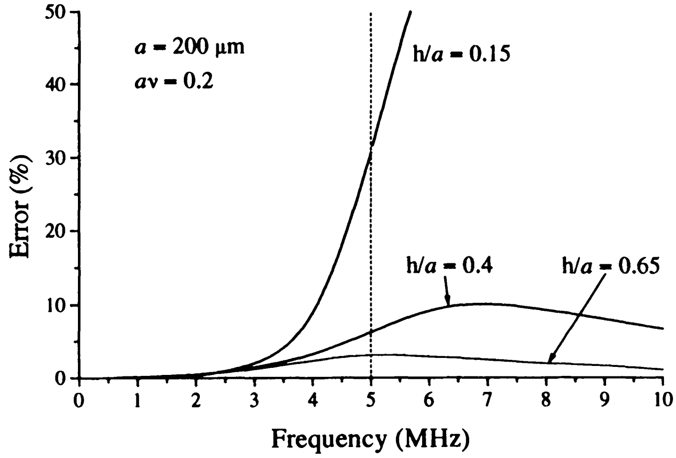


Figure 7. Error vs. angular frequency for three values of the layer thickness. The crack length is $a = 200$ μm, and the crack density is $a\nu = 0.2$.

spaces. The layered system is that considered in the previous figure. It should be stressed that the largest error is found where the sensitivity to the interfacial imperfection is expected to be the highest. Similar results are obtained for the other interfaces considered in Fig. 3.

## CONCLUDING REMARKS

The QSA boundary conditions were derived for a cracked interface from first principles. The derivation lead to expressions for the interfacial stiffness constants in terms of geometrical properties of the crack distribution, as well as of the micro-mechanics of the interfacial defects. In this work, the stiffness constants have been obtained under the assumption that the distributed cracks do not interact with each other. The extension of this modeling to include the interaction among first neighbors should be conceptually straightforward. Multiple crack interaction is expected to introduce cross terms in the boundary conditions.

The stiffness constants of an interface between a layer and a substrate have been shown to depend, in general, on the layer thickness. Therefore, these constants can no longer be regarded as intrinsic properties of the interface, and determined only by the imperfections, but the structural properties of the system should also be taken into account.

The relationship between the dependence of the stiffness constants on the layer thickness and the frequency dependence of the phase velocity of the lowest mode of the interface damage has been considered. It has been shown that ignoring the thickness dependence of the interfacial constants can cause errors up to 30 percent in the predicted values of the phase velocity.

## REFERENCES


1. J.M. Baik and R.B. Thompson, JNDE 4, 177 (1984).
2. M. Comninou, J. Appl. Mech. 44, 631 (1977).
3. P.B. Nagy, JNDE 11, 127 (1992).